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Race Car Tire Modeling: A Simplified Approach for Vehicle Dynamic Analysis and Real-Time Simulation

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ABSTRACT

An algorithm for tire data generation from on-track race car data to be used in vehicle stability and control analysis and/or lap time simulations is presented in this paper. The proposed algorithm allows analytical prediction of tire dynamic and kinematic quantities for a satisfactory evaluation of tire stiffness properties in the linear and non-linear range of operation. In particular, tire data related to Lateral Force vs. Slip Angle, Aligning Torque vs. Slip Angle, Braking (Driving) Force vs. Longitudinal Slip curves are made available by the algorithm. The resulting tire model responds to normal load, camber angle, and composite tire patch slip. Its longitudinal and lateral forces interact with an equivalent friction ellipse formulation.

Non-linear effects cannot be modeled in a frequency domain solution: a time domain solution is therefore used in examining tire dynamic behavior so that non-linear effects can be included in the analysis.

INTRODUCTION

Analysis and on-board instrumentation are normally used in design and tuning by professional racing teams. Specialized computer programs enable prediction of the stability and control of a race car given accurate chassis, tire and aero data. Lap time prediction, including details of performance around the circuit are approaching a level of sophistication useful for chassis tuning. These are theoretically possible but their achievement depends on acceptable model complexity, cost to use, available input data and the degree to which it is necessary to simulate the driver. Input data availability seems to be a critical point: in particular, lack of tire data makes it difficult to perform an accurate analysis in most cases. Mathematical models for the study and prediction of vehicle stability and control analysis are in fact only valid and efficient if an adequate submodel is available to describe mechanical behavior of tires in use, for road vehicles and race cars alike.

It is to be noted that the most commonly used tire models in the auto industry (Adams 2D, 3D tire models [Davis 1974, Negrut and Freeman 1994], F-Tire model [Gipser 1999], RMOD-K model [Oertel and Fandre 1999], SWIFT Tire model [Zegelaar 1998, Maurice1999, Pacejka 2002], VPG model) - developed for ride/handling simulations - hardly meet specific race car application requirements. On one hand, commercially available tire models generally require time consuming tests for tire model calibration and lengthy computer simulations. On the other hand, race car application requires both *accuracy* (e.g capability of describing actual tire response in any operating conditions) and *efficiency* (fast computer simulations). Auto industry tire models thus appear to be unsuitable for race car application.

Generating tire data from on-track race car data is a challenging but potentially successful approach to obtain useful tire information. The capability of describing actual tire response in any operating conditions is key in race car application, as the influence of operating conditions on car performance is so massive. The proposed algorithm therefore enables the real time assessment of tire stiffness properties so that tire data become available for stability and control analysis and/or lap time simulation.

ON-TRACK DATA ACQUISITION AND MODELING ANALYSIS APPROACH

The fundamental idea behind the proposed algorithm is to adopt a particular approach in modeling mechanical systems (vehicle, tire and suspension), thanks to the data acquisition and processing as performed by the specific application. A brief description of the approach and of its whys and hows is given below.

MODELING OF A SYSTEM IN A TRANSIENT STATE BY MEANS OF STEADY-STATE SYSTEM MODELS - A data acquisition system is fitted to the vehicle to provide user with all relevant kinematic quantities of interest in vehicle stability and control investigations (wheel steering angle and suspension motion, vehicle forward velocity (4 wheel speed sensors) and yawing velocity, center of gravity longitudinal and lateral acceleration). The above time domain quantities are sampled at custom set frequency rate (frequency rate range: 1-1000 Hz), measured at a particular time *t* and named *vehicle kinematic configuration at time t.*

The real vehicle (or a subsystem, e.g. a tire) is not in a steady state during the whole measured period (lap time typically). Consider a small sampling interval of a few kinematic configurations instead. Vehicle behavior can be assumed to be adherent to steady state operating conditions. Steady state system models may therefore prove accurate enough to describe continuously transient-state systems.

Virtually continuous vehicle monitoring thanks to the data acquisition system is thus key to treating non-steady systems like race car or race car tires with models for steady systems.

MODELING OF A DAMPED SYSTEM BY MEANS OF UNDAMPED SYSTEM MODELS - Tiny time intervals between the vehicle kinematic configuration measured at time *t* and the subsequent kinematic configuration at time $t+\Delta t$ make it pointless to consider system damping effects analytically. Unlike the traditional analysis method (i.e. integration of system motion equation(s) during a given interval starting from the assigned initial boundary conditions), analysis performed in race car environment allows an accurate system analysis by means of simpler models. Models for undamped systems can therefore be applied to damped systems like suspensions and tires.

Before moving on to the next section, a block diagram of the algorithm is provided below:

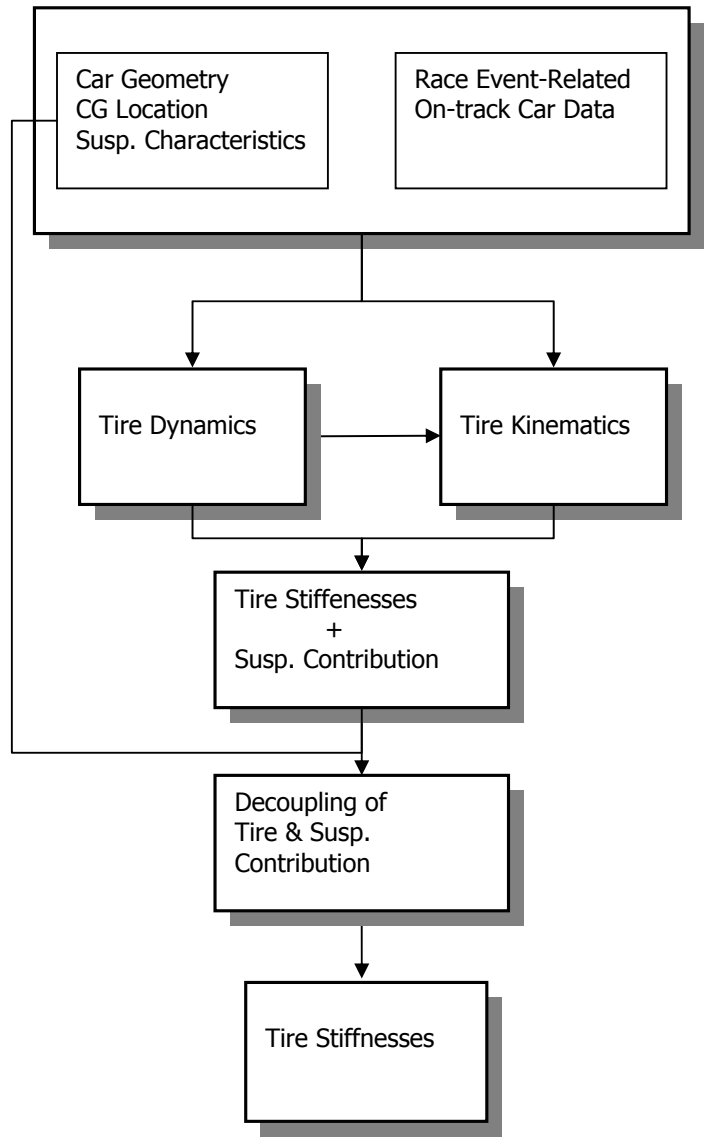

Figure 1 Algorithm Flowchart

TIRE DYNAMICS

The SAE Vehicle and Tire Axis Systems are adopted in this paper [13].

VEHICLE DYNAMICS (SINGLE TRACK MODEL) - Single track model dynamic equilibrium equations describe vehicle dynamics in terms of forces (Fxi, Fyi, Fzi) and moments (Mzi) on axle axes [12]. Significantly, these quantities are the result of on-track data processing and are therefore parametric of instantaneous tire operating conditions, i.e. boundary conditions (*internal b.c.:* tire slip angle, tire camber angle, tire

loaded radius, tire slip ratio, tire inflation air pressure; *external b.c.:* track surface and current friction coefficient at the tire-road interface, weather conditions).

VEHICLE DYNAMICS (REAL VEHICLE) – Considering previously acquired vehicle dynamics, tire loads F_{zij} can be evaluated by means of a formulation of total lateral load transfer distribution (for steady turn and tire deflection rates included in the front and rear roll rate values) contained in [11].

Forces on single tires will now be considered assuming the axle load as the only dynamic input on the axle. The remaining axle dynamic quantities can thus be expressed as a function of F_{zi}. This approach to axle (tire) dynamics might seem simplistic in applications other than the proposed one. The approach is nevertheless justified because forces and moments on tire axes (dynamic quantities) are the result of on-track data processing (said quantities are in fact parametric in boundary conditions) as seen previously. The relation between each axle load (F_{zi}) and axle forces and moments (F_{xi}, F_{yi}, M_{zi}) can be described by polynomial functions fitted to experimental data:

$$F_{xi} = a_{xi} F_{zi}^3 + b_{xi} F_{zi}^2 + c_{xi} F_{zi} + d_{xi} \quad (1)$$

$$F_{yi} = a_{yi} F_{zi}^3 + b_{yi} F_{zi}^2 + c_{yi} F_{zi} + d_{yi} \quad (2)$$

$$M_{zi} = a_{Mzi} F_{zi}^3 + b_{Mzi} F_{zi}^2 + c_{Mzi} F_{zi} + d_{Mzi} \quad (3)$$

Any degree of polynomial could be used. 3rd order polynomials have been used as an example.

For convenience, matrix notation is adopted. The following matrices relate to (1), (2), and (3) respectively:

$$\begin{bmatrix} F_{xi} \\ F_{yi} \\ M_{zi} \\ 0 \end{bmatrix} = \begin{bmatrix} a_{xi} & b_{xi} & c_{xi} & d_{xi} \\ a_{yi} & b_{yi} & c_{yi} & d_{yi} \\ a_{Mzi} & b_{Mzi} & c_{Mzi} & d_{Mzi} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_{zi}^3 \\ F_{zi}^2 \\ F_{zi} \\ 1 \end{bmatrix} \quad (4)$$

The forces and moments on i-axle matrix (left hand side), the F_{zi} parameter coefficient matrix and the F_{zi} parameter power matrix (right hand side) are defined in equation (4).

Based on the hypothesis discussed above on axle and tire loads, coefficients of equation (1), (2) and (3) are weighted in function of F_{zij} to evaluate the forces and moments on the tire axes.

Weight is defined as:

$$t_{ij} = \frac{F_{zij}}{F_{zi}} \quad (5)$$

The matrix of weighted parameter coefficients is obtained by simply multiplying t_{ij} by the matrix of parameter coefficients:

$$\begin{bmatrix} a_{xij} & b_{xij} & c_{xij} & d_{xij} \\ a_{yij} & b_{yij} & c_{yij} & d_{yij} \\ a_{Mzij} & b_{Mzij} & c_{Mzij} & d_{Mzij} \\ 0 & 0 & 0 & 0 \end{bmatrix} = t_{ij} \begin{bmatrix} a_{xi} & b_{xi} & c_{xi} & d_{xi} \\ a_{yi} & b_{yi} & c_{yi} & d_{yi} \\ a_{Mzi} & b_{Mzi} & c_{Mzi} & d_{Mzi} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (6)$$

The tire dynamic quantities vector is therefore expressed as follows:

$$\begin{bmatrix} F_{xij} \\ F_{yij} \\ M_{zij} \\ 0 \end{bmatrix} = \begin{bmatrix} a_{xij} & b_{xij} & c_{xij} & d_{xij} \\ a_{yij} & b_{yij} & c_{yij} & d_{yij} \\ a_{Mzij} & b_{Mzij} & c_{Mzij} & d_{Mzij} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_{zi}^3 \\ F_{zi}^2 \\ F_{zi} \\ 1 \end{bmatrix} \quad (7)$$

In other words, the vector obtained describes the forces and moments acting on the tire axes.

TIRE KINEMATICS

VEHICLE KINEMATICS (SINGLE TRACK MODEL) – Measures obtained through data acquisition systems are affected by unavoidable inaccuracy, which makes it difficult to evaluate vehicle lateral velocity and consequently makes the traditional single track model kinematic expression of slip angles virtually ineffective [12]. A new formulation was therefore adopted in acquiring vehicle kinematics and tire slip angles in particular. The new formulation is based on the traditional single-track model vehicle representation.

The process that led to the proposed formulation is summarized below as are the results of relevant experiments.

A New Formulation of Slip Angles for the Single Track Model. A new formulation for vehicle single track model tire slip angles is proposed.

The following results from the ratio between the expression for vehicle front axle (linearized) cornering stiffness and the one for rear axle (linearized) cornering stiffness:

$$\frac{\alpha_1}{\alpha_2} = \frac{F_{y1} C_{y2}}{F_{y2}\ C_{y1}} \qquad (8)$$

The following equation is obtained by replacing F_{y1}, F_{y2} with the relevant expressions in function of lateral friction coefficients and tire loads:

$$\frac{\alpha_1}{\alpha_2} = \frac{\mu_{y1}\ F_{z1}\ C_{y2}}{\mu_{y2}\ C_{y1}\ F_{z2}} \qquad (9)$$

Equation (9) contains cornering stiffness coefficients (right hand side) and can be written as follows:

$$\frac{\alpha_1}{\alpha_2} = \frac{\mu_{y1}\ CK_2}{\mu_{y2}\ CK_1} \qquad (10)$$

Considering the general case of tires with different aspect ratios on each axle as often occurs in racing cars, it can reasonably be assumed for the front and rear tire operating cornering stiffness coefficient values that:

$$CK_1 \approx CK_2 \qquad (11)$$

Finally, re-writing (10) in the light of (11) leads to:

$$\frac{\alpha_1}{\alpha_2} \approx \frac{\mu_{y1}}{\mu_{y2}} \qquad (12)$$

The above equation establishes a proportional relation between front axle and rear axle slip angles and their lateral friction coefficients.

Equation (12) and the kinematic expression of the single-track model front wheel steer angle δ [12] provide a satisfactory evaluation of axle slip angles as a function of vehicle dynamic, kinematic and mechanical quantities:

$$\alpha_1 \approx \frac{\mu_{y1}}{\mu_{y2}-\mu_{y1}}\ (\delta - \delta_{Ack}) \qquad (13)$$

$$\alpha_2 \approx \frac{\mu_{y2}}{\mu_{y2}-\mu_{y1}}\ (\delta - \delta_{Ack}) \qquad (14)$$

Note - the traditional expression for slip angles can now be used to determine the vehicle lateral velocity. The following equation is obtained considering the expression for rear axle slip angle (linearized, for non-steering rear axle vehicles) used in reverse for vehicle lateral velocity:

$$v = r\,b - \alpha_2\,u$$

Where α_2 appearing in the last expression is clearly obtained from equation (14);

A New Formulation of Slip Angles for the Single Track Model – Experimental Validation. Figure 2 shows an estimate of vehicle lateral velocity as obtained from on-track Formula 1 car data acquired at Hockenhiem track (Germany) by means of the above described data acquisition system (page 2). Experimental results supporting the proposed formulation are illustrated by the remaining figures. Said results were obtained by processing on-track racing go-kart (100 cc) data collected in Jesolo (Italy). The data acquisition system fitted to the vehicle provided wheel steering angle, vehicle forward velocity (2 wheel speed sensors) and center of gravity lateral acceleration.

Figure 2 shows how difficult it is to obtain a useful estimate of vehicle lateral velocity even if cutting-edge F1 car data acquisition system sensors are used. The Poisson formula $a_x = \dot{u} - vr$ has been reversed to calculate vehicle lateral velocity. Figure 3 shows results obtained for vehicle lateral velocity estimated in accordance with the proposed formulation. Figure 4 shows a simple but effective accuracy test for the vehicle lateral velocity estimate, consisting in a comparison between the go-kart longitudinal acceleration evaluated as vehicle forward velocity first derivative and the go-kart longitudinal acceleration obtained by using the Poisson formula ($a_x = \dot{u} - vr$, where u corresponds to measured forward velocity, r corresponds to yawing velocity - here estimated in steady state conditions (no vehicle yawing velocity direct measure available) - v corresponds to vehicle lateral velocity and is evaluated according to the proposed formulation). Interestingly, the vehicle longitudinal acceleration value obtained through a direct measure (accelerometer) will be somewhat higher (absolute value) than the value obtained from the vehicle forward velocity first derivative. The former will in fact be closer to the Poisson formula estimate. Finally, Figure 5 shows a realistic estimate of go-kart axle slip angles as obtained through the proposed formulation.

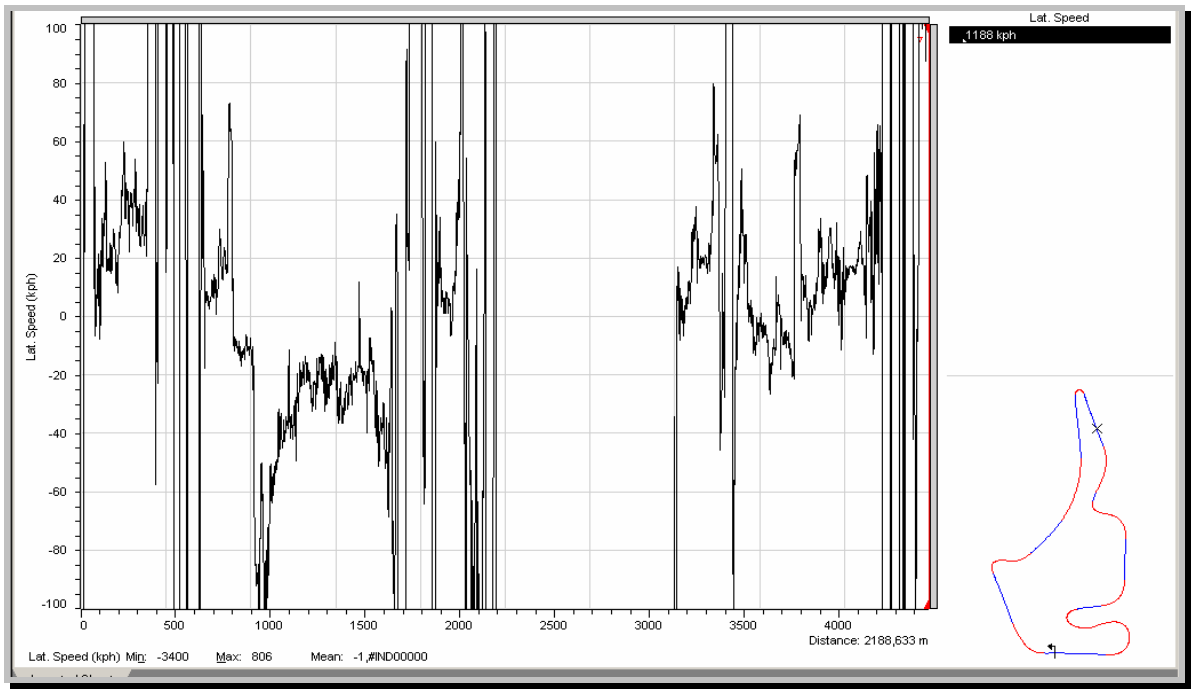

Figure 2 F1 Car Vehicle Lateral Velocity (km/h) vs. Distance (m), obtained by using Poisson formula ($a_x = \dot{u} - vr$) in reverse for vehicle lateral velocity. Note that a moving average filter is applied to the obtained signal.

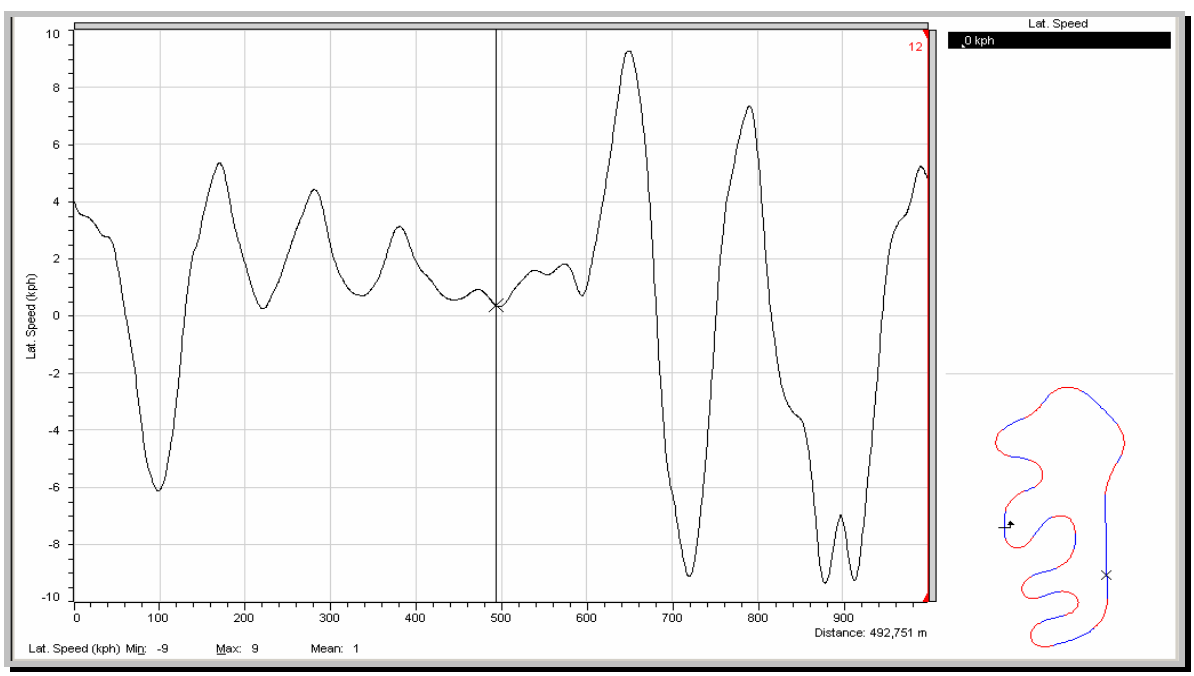

Figure 3 Go-kart Vehicle Lateral Velocity (km/h) vs. Distance (m), obtained by using the proposed formulation

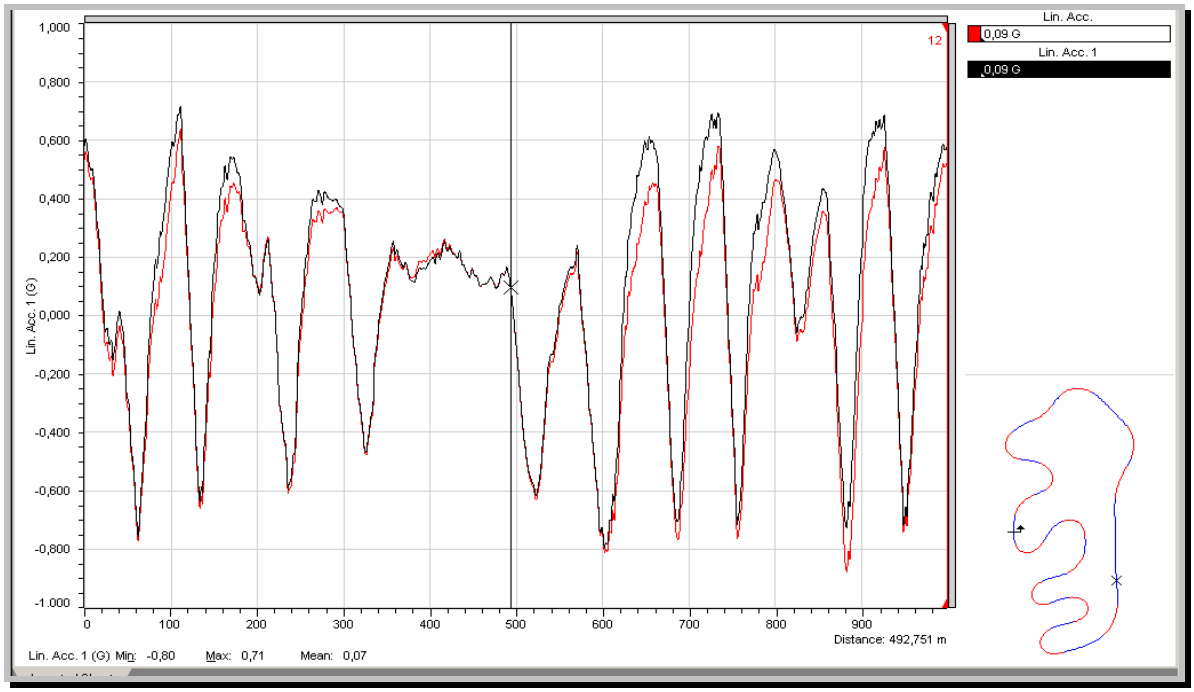

Figure 4 Lower red trace: Go-kart Longitudinal Acceleration (g) (vehicle forward velocity first derivative) vs. Distance (m), upper black trace: Go-kart Longitudinal Acceleration (g) (obtained by using Poisson formula ($a_x = \dot{u} - vr$)) vs. Distance (m)

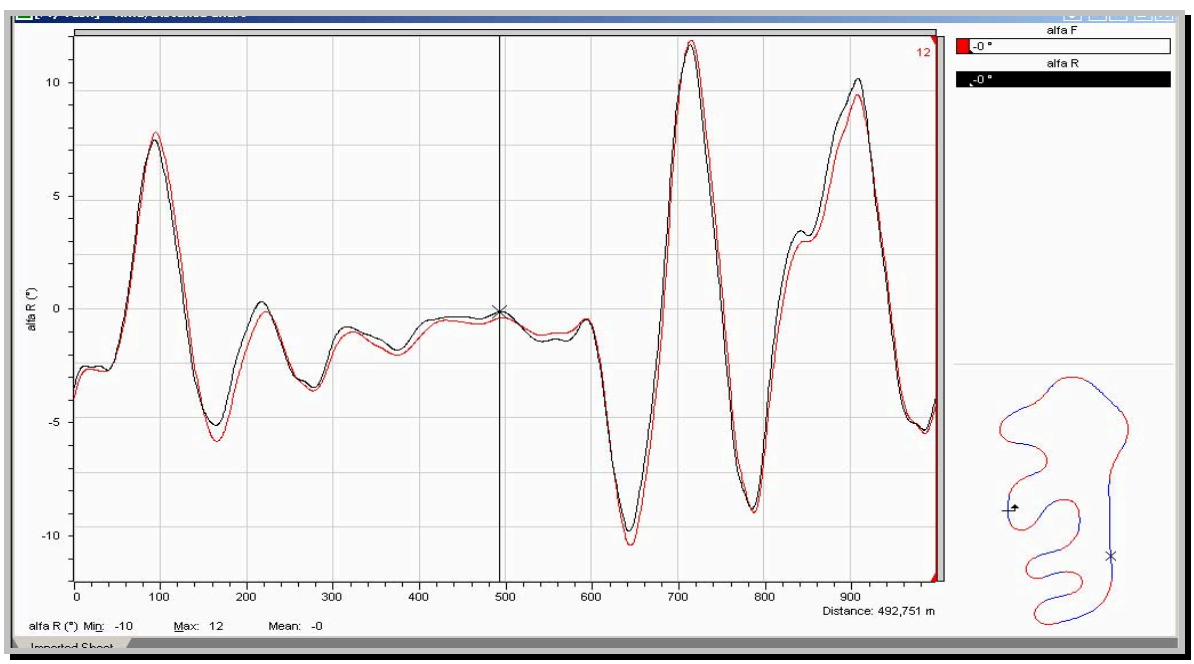

Figure 5 Go-kart Front Axle Slip Angle (deg) (red trace) and Rear Axle Slip Angle (deg) (black trace) vs. Distance (m)

VEHICLE KINEMATICS (REAL VEHICLE) – Experimental results show the validity of equations (13) and (14), whose scope of application can now be extended to single tires.

Slip angles for the 4 tires are therefore expressed as follows:

$$\alpha_{1j} \approx \frac{(\mu_y)_{1j}}{(\mu_y)_{2j}-(\mu_y)_{1j}} (\delta - \delta_{Ack.}) \quad (15)$$

$$\alpha_{2j} \approx \frac{(\mu_y)_{2j}}{(\mu_y)_{2j}-(\mu_y)_{1j}} (\delta - \delta_{Ack.}) \quad (16)$$

LONGITUDINAL SLIP – A thorough survey of tire dynamic and kinematic quantities - here used to obtain tire data - must include assessment of the longitudinal slip. A first rough estimate of the longitudinal slip can be obtained basing on the definition contained in [13], as tire angular and forward velocity are acquired from on-track data and slip angles are provided by processing on-track data through the proposed formulation (equation (15) and (16)).

SUSPENSION CONTRIBUTION TO TIRE STIFFNESS

The following has been previously assumed regarding forces on vehicle axles and on single tires:

- vehicle chassis and suspensions are infinitely rigid (single track model analysis)
- tire deflection rates are included in the front and rear roll rate values (real vehicle analysis)

In acquiring tire dynamics and kinematics, tire and suspension have been considered as an assembly (vehicle chassis considered infinitely rigid for a race car), and their individual contribution has been neglected.

Information about tire-suspension assembly-related dynamic and kinematic quantities is now available in detail. Tire stiffness properties in the linear and non-linear range of operation (not filtered for the suspension contribution) can be obtained as follows:

$$(K_x)_{ij} = \frac{\partial F_{xij}}{\partial s_{ij}} \quad (17)$$

$$(K_y)_{ij} = -\frac{\partial F_{yij}}{\partial \alpha_{ij}} \quad (18)$$

$$(K_{xy})_{ij} = \frac{\partial M_{zij}}{\partial \alpha_{ij}} \quad (19)$$

DECOUPLING OF TIRE AND SUSPENSION STIFFNESSES

A procedure for decoupling the tire and suspension stiffness contributions is proposed below. No specific suspension geometry is considered.

The tire and suspension assembly is now modeled with an equivalent mechanical system made up of 3 spring-mass-damper systems in series. The 3 spring-mass-damper system springs in series (generally having no parallel axes) depict tire, suspension spring and antiroll bar stiffness respectively (suspension spring and antiroll bar stiffness assigned and suspension arm stiffness remarkably higher than tire, spring and antiroll bar stiffness).

Suspension and tire damping effects will not be factored in as a damped system modeling approach has been adopted.

Tire cornering stiffness (referring to the slope of the lateral force curve) is calculated below as an example. The same procedure can be applied to the aligning stiffness and braking (driving) stiffness with the necessary changes.

As a first step, suspension spring stiffness $(K_s)_{ij}$ and motion $(x_s)_{ij}$ on spring axes shall be expressed as stiffness and motion on the tire axes.

$$[(K_s)_{ij}]_t = R_{ij} * T_{ij} * (K_s)_{ij} \quad (20)$$

$$[(x_s)_{ij}]_t = R_{ij} * T_{ij} * (x_s)_{ij} \quad (21)$$

The same applies to the antiroll bar.

Regarding dimensional aspects, stiffness components of the tire-suspension assembly are as follows:

$$(C_y)_{ij} \rightarrow \frac{kg}{\deg}$$

$$(K_s)_{ij} \rightarrow \frac{kg}{mm}$$

$$(K_b)_{ij} \rightarrow \frac{kg}{mm}$$

For convenience, a single measure is adopted for spring and antiroll bar stiffnesses and tire cornering stiffness (unknown quantity).

Spring stiffness can be expressed in $\frac{kg}{\deg}$ as follows:

tire slip angle α_{ij} is expressed in function of suspension motion $[(x_s)_{ij}]_t$; the relation between slip angle and suspension motion can be described by polynomial functions fitted to experimental data:

$$\alpha_{ij} = (a_s)_{ij}[(x_s)_{ij}]_t^3 + (b_s)_{ij}[(x_s)_{ij}]_t^2 + (c_s)_{ij}[(x_s)_{ij}]_t + (d_s)_{ij} \quad (22)$$

Spring stiffness can be expressed in the new measure as follows:

$$[(K_s)_{ij}]_t^{\alpha} = \left(\frac{d\alpha_{ij}}{d(x_s)_{ij}}\right)^{-1} [(K_s)_{ij}]_t \quad (23)$$

The same applies to the antiroll bar.

All stiffnesses are now expressed as stiffnesses on tire axes and a single measure has been adopted. The following equation therefore makes sense:

$$(K_y)_{ij}^{-1} = (C_y)_{ij}^{-1} + [(K_s)_{ij}]_t^{\alpha^{-1}} + [(K_b)_{ij}]_t^{\alpha^{-1}} \quad (24)$$

Tire cornering stiffness can be worked out from the above equation.

CONCLUSIONS

The tire modeling tool presented here was developed to meet specific racing application requirements in terms of accuracy and efficiency. The proposed algorithm provides just a rough estimate and can of course be improved by adopting fine-tuned models for the generation of tire kinematics and dynamics, although that may reduce its efficacy. This paper has nevertheless provided an outline for a viable approach to race car tire data generation.

Experimental results obtained through the proposed algorithm are not available yet. A Simulink-type model is currently under development to provide an input for stability and control analysis and/or lap time simulations and thus enable testing.

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LIST OF SYMBOLS

- a_x = vehicle center of gravity longitudinal acceleration
- b = rear axle distance from center of gravity
- C_{yi} = i-th axle cornering stiffness
- $(C_y)_{ij}$ =i-th – j-th tire cornering stiffness
- CK_i = i-th axle cornering stiffness coefficient
- F_{xi}, F_{yi}, F_{zi} = forces on i-th axle axes
- $F_{xij}, F_{yij}, F_{zij}$ = forces on i-th – j-th tire axes
- $(K_x)_{ij}$ = i-th – j-th tire braking (driving) stiffness not filtered for the suspension contribution (on tire axes)
- $(K_y)_{ij}$ = i-th – j-th tire cornering stiffness not filtered for the suspension contribution (on tire axes)
- $(K_{xy})_{ij}$ = i-th – j-th tire aligning stiffness not filtered for the suspension contribution (on tire axes)
- $(K_s)_{ij}$ = i-th – j-th suspension spring stiffness on spring axes
- $[(K_s)_{ij}]_t$ = i-th – j-th suspension spring stiffness on tire axes $\left(\frac{\text{kg}}{\text{mm}}\right)$
- $[(K_s)_{ij}]_t^{\alpha}$ = i-th – j-th suspension spring stiffness on tire axes $\left(\frac{\text{kg}}{\text{deg}}\right)$
- $[(K_b)_{ij}]_t^{\alpha}$ = i-th – j-th antiroll bar stiffness on tire axes $\left(\frac{\text{kg}}{\text{deg}}\right)$
- M_{zi} = i-th axle aligning moment
- M_{zij} = i-th – j-th tire aligning moment
- r = vehicle yawing velocity
- R_{ij} = i-th – j-th suspension spring axes to tire axes rotation matrix
- s_{ij} = i-th – j-th tire longitudinal slip
- t = typical time instant
- t_{ij} = weight
- T_{ij} = i-th – j-th suspension spring axes to tire axes translation matrix
- u = vehicle forward velocity
- v = vehicle lateral velocity
- $(x_s)_{ij}$ = i-th – j-th suspension spring motion on spring axes
- $[(x_s)_{ij}]_t$ = i-th – j-th suspension spring motion on tire axes

- α_i = i-th axle slip angle
- α_{ij} = i-th – j-th tire slip angle
- δ = front wheel steer angle
- $\delta_{Ack.}$ = Ackerman steer angle at front wheel
- μ_{yi} = i-th axle lateral friction coefficient
- $(\mu_y)_{ij}$ = i-th – j-th tire lateral friction coefficient

i= 1, 2	1= front	2= rear
j= R, L	R= right	L= left